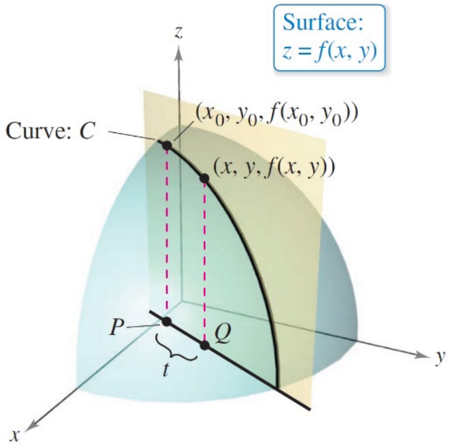
Directional Derivative 方向导数数(p915)

a surface: $z = f(x,y)$

a point: $P(x_0, y_0)$

a direction: $\vec{u} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$

To find the desired slope along direction $\vec{u}$, reduce the problem to two dimensions by intersecting the surface with a vertical plane passing through the point P and parallel to $\vec{u}$ as shown in the right Figure. This vertical plane intersects the surface to form a curve C. The slope of the surface at $(x_0, y_0, f(x_0, y_0))$ in the direction of $\vec{u}$ is defined as the slope of the curve C at that point.



The vertical plane used to form C intersects the xy-plane in a line L represented by the parametric equations

$$\begin{cases} x = x_0 + t\cos\theta \\ y = y_0 + t\sin\theta \end{cases}$$

Point P,Q have corresponding points on the surface

$(x_0, y_0, f(x_0, y_0))$ Point above P

$(x, y, f(x, y))$ Point above Q

We can get the slope of the secant line

$$\frac{f(x,y) - f(x_0,y_0)}{t} = \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}$$

Directional Derivative: can be obtained by letting t approach 0:

$$D_u f(x,y) = \lim_{t\to 0} \frac{f(x_0 + t\cos\theta, y_0 + t\sin\theta) - f(x_0, y_0)}{t}$$

If f is a differentiable function of x and y

$$D_u f(x,y) = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$\vec{u} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$$

Example 1: Find the directional derivative of $f(x,y) = 4 - x^2 - y^2/4$ at $(1,2)$ in the direction of $\vec{u} = \cos(\pi/3)\,\vec{\imath} + \sin\theta(\pi/3)\,\vec{\jmath}$.

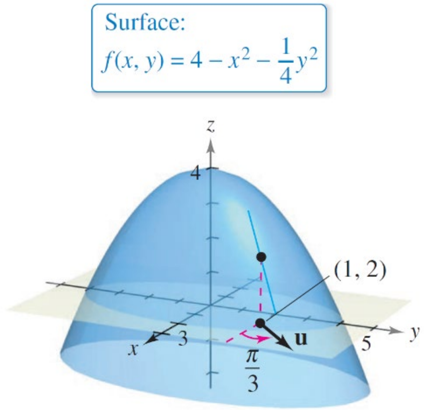
Solution:

$$\begin{cases} \dfrac{\partial f}{\partial x} = -2x \ \text{: continuous} \\ \dfrac{\partial f}{\partial y} = -\dfrac{y}{2} \ \text{: continuous} \\ f(x,y) \ : differentiable \end{cases}$$

$$D_u f(x,y) = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$= (-2x)\cos\theta - \frac{y}{2}\sin\theta$$

$$= (-2)\left(\frac{1}{2}\right) - \frac{2}{2}\left(\frac{\sqrt{3}}{2}\right) = -1 - \frac{\sqrt{3}}{2}$$



Example 2: Find the directional derivative of $f(x,y) = x^2\sin 2y$ at $(1, \pi/2)$ in the direction of $\vec{v} = 3\vec{\imath} - 4\vec{\jmath}$.

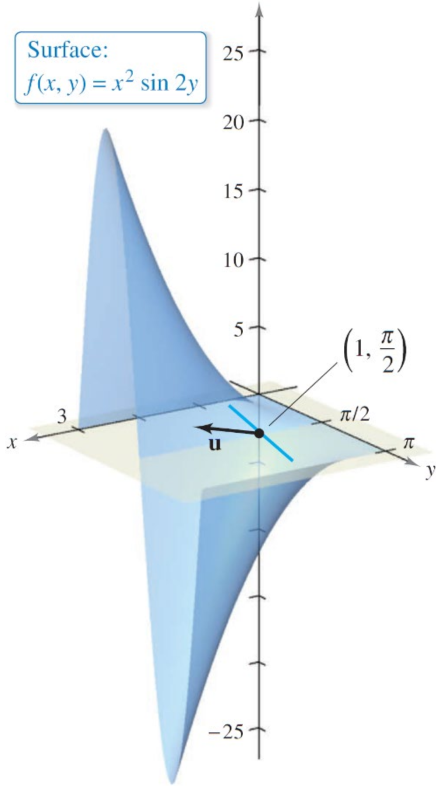
$$\begin{cases} \dfrac{\partial f}{\partial x} = 2x\sin 2y \ \text{: continuous} \\ \dfrac{\partial f}{\partial y} = 2x^2\cos 2y \text{: continuous} \\ f(x,y) \ : differentiable \end{cases}$$

$\cos\theta = 3/5, \quad \sin\theta = -4/5$

$$D_u f(x,y) = \frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta$$

$$= (2x\sin 2y)\cos\theta + (2x^2\cos 2y)\sin\theta$$

$$D_u f(1, \pi/2) = 2\sin\pi\left(\frac{3}{5}\right) + (2\cos\pi)\left(-\frac{4}{5}\right) = 8/5$$



The Gradient of a Function of Two Variables 梯度(p918)

Let $z = f(x, y)$ be a function of x and y such that $\partial z/\partial x$ and $\partial z/\partial y$ exist. Then the gradient of $f(x, y)$, denoted by $\vec{\nabla} f(x, y)$, is the vector

$$\boldsymbol{grad}\, f(x, y) = \vec{\nabla} f(x, y) = \frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j}$$

The symbol $\vec{\nabla} f$ is read as "del f."

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j}$$ 称为(二维)向量微分算子或 Nabla 算子

Example 1: Find the Gradient of $f(x, y) = y\ln x + xy^2$ at the point $(1,2)$.

Solution:

$$\frac{\partial f}{\partial x} = \frac{y}{x} + y^2$$

$$\frac{\partial f}{\partial y} = \ln x + 2xy$$

$$\vec{\nabla} f(x, y) = \frac{\partial z}{\partial x}\vec{i} + \frac{\partial z}{\partial y}\vec{j} = \left(\frac{y}{x} + y^2\right)\vec{i} + (\ln x + 2xy)\vec{j}$$

At the point $(1,2)$

$$\vec{\nabla} f(1,2) = \left(\frac{2}{1} + 2^2\right)\vec{i} + \left(\ln 1 + 2(1)(2)\right)\vec{j} = 6\left(\frac{y}{x} + y^2\right)\vec{i} + (\ln x + 2xy)\vec{j} + 4\vec{j}$$

Because the gradient of f is a vector, you can write the directional derivative of f in the direction of unit vector as

$$D_u f(x, y) = \vec{\nabla} f(x, y) \cdot (\cos\theta\,\vec{i} + \sin\theta\,\vec{j})$$

Alternative Form of the Directional Derivative

$$D_u f(x, y) = \vec{\nabla} f(x, y) \cdot \vec{u}$$

Properties of the Gradient

1. if $\vec{\nabla} f(x, y) = 0, then\ D_u f(x, y) = 0\ for\ all\ \vec{u}$
2. The direction of maximum increase of f is given by $\vec{\nabla} f(x, y)$

$$\{D_u f(x, y)\}_{max} = \left\|\vec{\nabla} f(x, y)\right\|$$

Example 2: The temperature in degrees Celsius on the surface of a metal plate is

$$T(x, y) = 20 - 4x^2 - y^2$$

where x and y are measured in centimeters. In what direction from $(2, -3)$ does the temperature increase most rapidly? What is this rate of increase?

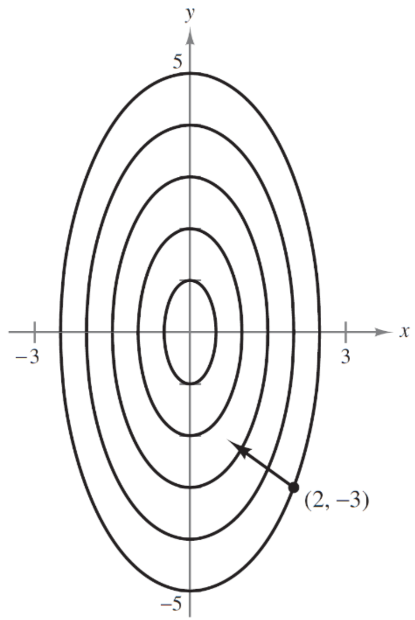
Solution: The gradient is

$$\vec{\nabla} T(x, y) = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j} = -8x\vec{i} - 2y\vec{j}$$

It follows that the direction of maximum increase is given by

$$\vec{\nabla} T(2, -3) = -16\vec{i} + 6\vec{j}$$

$$\left\|\vec{\nabla} T(2, -3)\right\| = \sqrt{256 + 36} = \sqrt{292}$$



Gradient Is Normal to Level Curves

If f is differentiable at (x_0, y_0) and $\vec{\nabla} f(x_0, y_0) \neq 0$ then $\vec{\nabla} f(x_0, y_0)$ is normal to the level curve through (x_0, y_0)

Example 2: A heat-seeking particle is located at the point $(2, -3)$ on a metal plate whose temperature at (x, y) is $T(x, y) = 20 - 4x^2 - y^2$. Find the path of the particle as it continuously moves in the direction of maximum temperature increase.

Solution: Let the path be represented by the position vector

$$\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$$

A tangent vector at each point $(x(t), y(t))$ is given by

$$\frac{d}{dt}\vec{r}(t) = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j}$$

Because the particle seeks maximum temperature increase, the directions of $d\vec{r}(t)/dt$ and $\vec{\nabla} T(x, y) = -8x\vec{i} - 2y\vec{j}$ are the same at each point on the path. So,

$$-8x = k\frac{dx}{dt} \quad \text{and} \quad -2y = k\frac{dy}{dt}$$

Where k depends on t.

$$\frac{dt}{k} = \frac{dx}{-8x}, \quad \frac{dt}{k} = \frac{dy}{-2y} \quad \Rightarrow \frac{dx}{-8x} = \frac{dy}{-2y} \quad \Rightarrow x = \frac{1}{81}y^4$$